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ON THE SPACE OF LINEAR HOMEOMORPHISMS OF A POLYHEDRAL N-CELL

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The author shows that the space of all the linear homeomorphisms of a triangulated n-cell into the n-dimensional euclidean space E^n has the structure of a fibre bundle with an open n-cell as its fibres. If the triangulation has exactly two interior vertices, the fibre bundle is homeomorphic to the product bundle $E^n \times E^n$. This improves a previous result of the author, which was obtained in answering a question raised by R.H. Bing and M. Starbird.

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1. Introduction and the main results

In the late 70's, R.H. Bing and Michael Starbird studied linear embeddings and isotopies in Euclidean spaces [1, 2, 9, 10]. They considered, among other things, the following problem. Let (D, T) be a closed 2-dimensional disk D with a triangulation T such that the triangulated disk is linearly embedded in the euclidean plane E^2. A homeomorphism $f: D \to D$ is called a *linear homeomorphism* of (D, T) if f is pointwise fixed on $\mathrm{Bd}(D)$ and is, in addition, linear on each simplex of T. An isotopy h_t between two linear homeomorphisms f and g of (D, T) is an ordinary isotopy between f and g with the extra condition that for each t, h_t is pointwise fixed in $\mathrm{Bd}(D)$ and is linear on each simplex of T. The problem considered by Bing and Starbird is this: under what conditions is a linear homeomorphism of (D, T) linearly isotopic to the identity map of D?

Since the linear homeomorphism of (D, T) are pointwise fixed on $\mathrm{Bd}(D)$, if T has only one interior vertex (i.e., a vertex in the interior of D), the problem is trivial. On the other hand, Bing and Starbird constructed an example of a 2-dimensional disk (D, T) in E^2 with three interior vertices such that there is a linear homeomorphism of (D, T) which is not isotopic to the identity map of D [1, Example 4.1]. This leads naturally to the question as what happens when T has exactly two interior vertices (see [2, p. 325])?

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We may look at this problem from a function space point of view. Let $\mathcal{H}(D, T)$ be the space of all the linear homeomorphisms of (D, T) under the compact open topology. An isotopy in D between two elements of $\mathcal{H}(D, T)$ will then correspond to a path in the space $\mathcal{H}(D, T)$. Bing and Starbird's question translates in this setting to the problem whether the space $\mathcal{H}(D, T)$ is pathwise connected. This question was answered by the author in a previous paper. He showed that the space $\mathcal{H}(D, T)$ is in fact contractible even when (D, T) is a triangulated n-dimensional cell in E^n with two interior vertices [5]. In this paper, we shall further improve this result and show that for a triangulated n-dimensional cell (D, T) in E^n with 2 interior vertices, the space $\mathcal{H}(D, T)$ is actually homeomorphic to the euclidean space E^{2n}. We shall prove this by establishing another result which may have some interest in its own right. For an arbitrary triangulated n-dimensional cell (D, T) linearly embedded in E^n, where T has k interior vertices, the space $\mathcal{H}(D, T)$ may be considered as a fibre bundle with an n-dimensional open cell as its fibre and an open subset of $E^{(k-1)n}$ as its base space. For the case when T has exactly two interior vertices, the base space for this fibre bundle is a bounded star-shaped open subset of E^n. In this case, the space $\mathcal{H}(D, T)$ is easily shown to be homeomorphic to E^{2n}.

2. Proof of the main results

To facilitate the proof, we need a few notations. For a simplicial complex K and a vertex v in K, we shall let $\mathrm{St}(v, K)$ and $\mathrm{Lk}(v, K)$ be the *open star* and the *link* of v in K, respectively. By a *polyhedral* $(n-1)$*-dimensional sphere* in E^n, we mean a simplicial complex Σ which is linearly embedded in E^n and whose underlying space $|\Sigma|$ is homeomorphic to the $(n-1)$-sphere S^{n-1}. Given such a Σ, we shall let $[\Sigma]$ be the union of $|\Sigma|$ with the bounded component of $E^n - |\Sigma|$. A polyhedral $(n-1)$-sphere Σ in E^n is said to be *star-shaped* if there is a point $s \in \mathrm{Int}[\Sigma]$ such that for every $x \in |\Sigma|$, the half open line segment $[s, x[\ (= \{p \in E^n \mid p = (1-t)s + tx, 0 \leq t < 1\})$ lies entirely in $\mathrm{Int}[\Sigma]$. We say that such a point s *can see* every point of Σ.

Notation 1. Let Σ be a polyhedral $(n-1)$-dimensional sphere in E^n. For each $(n-1)$-simplex σ of Σ, let $H_\sigma(\Sigma)$ be the unique open half-space of E^n such that:

(1) The simplex σ lies in $\mathrm{Bd}(H_\sigma(\Sigma))$, and

(2) $\bar{H}_\sigma(\Sigma)$ (the closed half-space) contains a neighborhood, with respect to the space $[\Sigma]$, of the open $(n-1)$-simplex σ.

Definition 2. Let Σ be a polyhedral $(n-1)$-dimensional sphere in E^n, we define the *core* of Σ, denoted by $\mathrm{cor}(\Sigma)$, to be the set

$$\mathrm{cor}(\Sigma) = \bigcap \{H_\sigma(\Sigma) \mid \sigma \text{ is an } (n-1)\text{-simplex of } \Sigma\}.$$

We shall also let $\overline{\mathrm{cor}}(\Sigma)$ be the intersection of the corresponding closed half-spaces. Note that $\mathrm{Int}(\overline{\mathrm{cor}}(\Sigma)) = \mathrm{cor}(\Sigma)$.

Remark. For any polyhedral $(n-1)$-sphere Σ in E^n, $\operatorname{cor}(\Sigma)$ is always a bounded open convex subset of the open cell $\operatorname{Int}[\Sigma]$. The set $\operatorname{cor}(\Sigma)$ is nonempty if and only if the polyhedral sphere Σ is star-shaped, and $\operatorname{cor}(\Sigma)$ consists of exactly those points of $\operatorname{Int}[\Sigma]$ which can see every point of Σ.

Now, let (D, T) be an n-dimensional closed cell D together with a triangulation T of k interior vertices such that (D, T) is linearly embedded in the euclidean space E^n. Let $\mathscr{H}(D, T)$ be the space of the linear homeomorphisms of (D, T) as defined in Section 1. Note that the compact open topology on $\mathscr{H}(D, T)$ is metrizable, say with a metric

$$\gamma(f, g) = \max\{d(f(v), g(v)) \mid v \text{ an interior vertex of } T\},$$

where d is the sup-metric on E^n (i.e., $d(x, y) = \operatorname{Sup}_{i=1}^n |x_i - y_i|$ for each $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E^n).

Remark. For each element $f \in \mathscr{H}(D, T)$, $f(T)$ is also a triangulation of D and for each interior vertex v_0 of T, $\operatorname{cor}(\operatorname{Lk}(f(v_0), f(T)))$ is a nonempty open bounded convex subset of E^n. The set is nonempty because the vertex $f(v_0)$, must lie in the set $\operatorname{cor}(\operatorname{Lk}(f(v_0), f(T)))$. Conversely, for each point $x \in \operatorname{cor}(\operatorname{Lk}(f(v_0), f(T)))$, there exists a unique $g \in \mathscr{H}(D, T)$ such that $g(v) = f(v)$ for each vertex $v \neq v_0$ of T and $g(v_0) = x$.

Notation 3. For any interior vertex v_0 of T, let $\mathscr{H}_{v_0}(D, T)$ be the subspace of $\mathscr{H}(D, T)$ consisting of all the linear homeomorphisms f of (D, T) such that $f(v_0)$ is located at the centroid of the set $\operatorname{cor}(\operatorname{Lk}(f(v_0), f(T)))$. We shall also define a map

$$\pi_{v_0}\colon \mathscr{H}(D, T) \to \mathscr{H}_{v_0}(D, T)$$

by letting $\pi_{v_0}(g)$ be the linear homeomorphism of (D, T) which agrees with g on all the vertices $v \neq v_0$ of T and $\pi_{v_0}(g)(v_0) =$ the centroid of the set $\operatorname{cor}(\operatorname{Lk}(g(v_0), g(T)))$.

Lemma 4. *The space $\mathscr{H}_{v_0}(D, T)$ is homeomorphic to an open subset of $E^{(k-1)n}$, where k is the number of interior vertices of T.*

Proof. Let T_0 be the subcomplex of T consisting of all the simplices of T which does not have v_0 as a vertex and let $D_0 = D - \operatorname{St}(v_0, T)$, then T_0 is a triangulation of D_0. Let $\mathscr{L}(D_0, T_0)$ be the set of all the linear maps (not necessarily homeomorphisms) $f\colon D_0 \to E^2$ such that f is pointwise fixed on $\operatorname{Bd}(D)$. $\mathscr{L}(D_0, T_0)$ will be considered as a topological space under the compact open topology. Note that if we assign an ordering to those vertices of T_0 which do not lie on $\operatorname{Bd}(D)$, say $v_1, v_2, \ldots, v_{k-1}$, then there is an one-to-one correspondence between the set $\mathscr{L}(D_0, T_0)$ and $E^{(k-1)n}$ with each element f of $\mathscr{L}(D_0, T_0)$ corresponding to the point $(f(v_1), f(v_2), \ldots, f(v_{k-1}))$ of $E^{(k-1)n}$. This correspondence is in fact an isometry if

$\mathscr{L}(D_0, T_0)$ is given the metric γ described in a previous remark and $E^{(k-1)n}$ is given the sup-metric. Thus, $\mathscr{L}(D_0, T_0)$ is identified with $E^{(k-1)n}$. Observe that the space $\mathscr{H}_{v_0}(D, T)$ corresponds to a subset of $\mathscr{L}(D_0, T_0)$, the subset consisting of those linear maps $f: D_0 \to E^n$ such that f is an embedding of D_0 into D and the set $\text{cor}(f(Lk(v_0, T)))$ is nonempty. Therefore, we need only show that $\mathscr{H}_{v_0}(D, T)$, considered as a subset of $\mathscr{L}(D_0, T_0)$, is open in $\mathscr{L}(D_0, T_0)$.

Let any $f \in \mathscr{H}_{v_0}(D, T)$ be given. We need to show that any element g of $\mathscr{L}(D_0, T_0)$ will also be in $\mathscr{H}_{v_0}(D, T)$ if it is sufficiently close to f under the metric γ. First note that the set $\text{cor}(f(\text{Lk}(v_0, T)))$ is the intersection of a finite collection of open half spaces in E^n. If this collection has a nonempty intersection, then under a sufficiently small perturbation for each half space in the collection, the new collection can again be made to have a nonempty intersection. Thus, we can fix a neighborhood $\mathscr{n}$ of f in $\mathscr{L}(D_0, T_0)$ such that for any g in $\mathscr{n}$, the set $\text{cor}(g(\text{Lk}(v_0, T))$ is nonempty. Note that each of such g has a unique extension to a linear map of (D, T) into E^n (by letting $g(v_0)$ be the centroid of the set $\text{cor}(g(\text{Lk}(v_0, T)))$. Now, for any $g \in \mathscr{n}$ sufficiently close to f, g will at least be locally one-to-one. This follows from a theorem of Whitehead [12] that a sufficiently close C^1 approximation to an immersion from a complex into a manifold is itself an immersion (also see [8, Theorem 8.8]). But being locally one-to-one, a linear map of (D, T) into E^n, which is pointwise fixed on $\text{Bd}(D)$, has to be an embedding of D onto D (see [6, Theorem 3.1]). Thus, choosing a smaller neighborhood $\mathscr{n}_0$ of f in $\mathscr{n}$ if necessary, we may conclude that all the elements $g \in \mathscr{n}_0$ belong to $\mathscr{H}_{v_0}(D, T)$. Thus, $\mathscr{H}_{v_0}(D, T)$ is an open subset of $\mathscr{L}(D_0, T_0)$.

Theorem A. *Let Δ^n be an open rectilinear n-simplex in E^n, (D, T) be a triangulated closed n-cell in E^n, and v_0 be an arbitrary interior vertex of T. Then $\pi_{v_0}: \mathscr{H}(D, T) \to \mathscr{H}_{v_0}(D, T)$ is a fibre bundle with fibre Δ^n and the structure group $G_0(\Delta^n)$, the group of all the orientation preserving homeomorphisms of Δ^n.*

Proof. First note that for each $f \in \mathscr{H}_{v_0}(D, T)$, $\pi_{v_0}^{-1}(f)$ consists of all the elements $g \in \mathscr{H}(D, T)$ such that $g(v) = f(v)$ for all the interior vertices v of T that is different from v_0, and $g(v_0) \in \text{cor}(\text{Lk}(f(v_0), f(T)))$. From the remark preceding to Notation 3, the set $\pi_{v_0}^{-1}(f)$ can be identified with the set $\text{cor}(\text{Lk}(f(v_0), f(T)))$ with each $g \in \pi_{v_0}^{-1}(f)$ being identified with the point $g(v_0)$. This identification is in fact an isometry with the metric γ in $\mathscr{H}(D, T)$ and the sup-metric in E^n. Thus $\pi_{v_0}^{-1}(f)$ is an open n-cell in E^n for each $f \in \mathscr{H}_{v_0}(D, T)$.

To show the bundle is locally trivial, for a given $f \in \mathscr{H}_{v_0}(D, T)$, we may assume that, by a translation and a contraction if necessary, a copy of the closed n-simplex $\bar{\Delta}^n$ is contained in $\text{cor}(\text{Lk}(f(v_0), f(T)))$. Now, choose an open neighborhood $\mathscr{U}$ of f in $\mathscr{H}_{v_0}(D, T)$ such that for each $g \in \mathscr{U}$, the set $\text{cor}(\text{Lk}(g(v_0), g(T)))$ contains $\bar{\Delta}_n$.

It is well known in the theory of convex sets that for each $g \in \mathscr{U}$, the set $\overline{\text{cor}}(\text{Lk}(g(v_0), g(T)))$, being a bounded nonempty intersection of a finite collection of closed half-spaces, is the convex hull of a finite set of points, the extreme points of $\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T)))$ (see e.g. [3, p. 29]). Note that the set $\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T)))$ has a natural structure as a rectilinear cell complex in the sense of [8, p. 74], where the complex has the extreme points of $\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T)))$ as its vertices, the set $\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T)))$ as its n-cell, and σ is one of its k-cells for $0 \leq k < n$ if and only if σ is the convex hull of a set of extreme points $e_1, e_2, \ldots, e_m$ of $\overline{\mathrm{cor}}(\mathrm{Lk}(g(n_0), g(T)))$ such that there exist $n-k$ $(n-1)$-simplices $\tau_1, \tau_2, \ldots, \tau_{n-k}$ in $\mathrm{Lk}(g(v_0), g(T))$ with the properties that:

(1) if H_i is the hyperplane in E^n determined by τ_i for each $i = 1, 2, \ldots, n-k$, the hyperplanes $H_1, H_2, \ldots, H_{n-k}$ are in general positions.

(2) $\{e_1, e_2, \ldots, e_m\} \subset \bigcap_{i=1}^{n-k} H_i$.

Let $h_{\mathscr{U}}(g)$ be the radial projection towards the barycenter of $\bar{\Delta}^n$ from $\mathrm{Bd}(\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T))))$ onto $\mathrm{Bd}(\bar{\Delta}^n)$. Since $h_{\mathscr{U}}(g)$ locally carries rectilinear cells transversally onto rectilinear cells, it is a homeomorphism of $\mathrm{Bd}(\overline{\mathrm{cor}}(\mathrm{Lk}(g(v_0), g(T))))$ onto $\mathrm{Bd}(\bar{\Delta}^n)$, a linear radial extension with respect to the barycenter of $\bar{\Delta}^n$ yields a homeomorphism (still denoted by $h_{\mathscr{U}}(g)$) from $\mathrm{cor}(\mathrm{Lk}(g(v_0), g(T)))$ onto Δ^n.

Now, let $\Phi_{\mathscr{U}} : \pi_{v_0}^{-1}(\mathscr{U}) \to \mathscr{U} \times \Delta^n$ be defined by $\Phi_{\mathscr{U}}(\bar{g}) = (\pi_{v_0}(\bar{g}), h_{\mathscr{U}}(\pi_{v_0}(\bar{g}))(\bar{g}(v_0)))$. It is not difficult to show that $\Phi_{\mathscr{U}}$ is a fibre-preserving homeomorphism of $\pi_{v_0}^{-1}(\mathscr{U})$ onto $\mathscr{U} \times \Delta^n$, and for any two such open neighborhoods $\mathscr{U}$ and $\mathscr{V}$ in $\mathscr{H}_{v_0}(D, T)$, the composite $h_{\mathscr{U}} \circ h_{\mathscr{V}}^{-1} : \mathscr{U} \cap \mathscr{V} \to G_0(\Delta^n)$ is continuous. Thus, $(\mathscr{H}(D, T), \pi_{v_0}, \mathscr{H}_{v_0}(D, T), \Delta^n, G_0(\Delta^n))$ is in fact a fibre bundle in the sense of Steenrod. □

Theorem B. *Let (D, T) be a triangulated closed n-cell in E^n. If T has only two interior vertices, the space $\mathscr{H}(D, T)$ is homeomorphic to E^{2n}.*

Proof. Fix an interior vertex v_0 of T. In a previous paper, the author has already shown that the space $\mathscr{H}_{v_0}(D, T)$ is star-shaped [5, pp. 234–236], and is therefore, contractible. It then follows from standard arguments in fibre bundle theory that

$$\pi_{v_0}: \mathscr{H}(D, T) \to \mathscr{H}_{v_0}(D, T)$$

is equivalent to the product bundle $\mathscr{H}_{v_0}(D, T) \times \Delta^n \to \mathscr{H}_{v_0}(D, T)$ (use [7] and [11, § 11.5]). But $\mathscr{H}_{v_0}(D, T)$ is also an open subset of E^n by Lemma 4. It is clearly a bounded open subset of E^n, for under the identification described in the proof of Lemma 4, $\mathscr{H}_{v_0}(D, T)$ is identified with a subset of D. Thus, $\mathscr{H}_{v_0}(D, T)$, being a bounded star-shaped open subset of E^n, is homeomorphic to an n-cell via a radial projection, and therefore, is homeomorphic to E^n. Hence, $\mathscr{H}(D, T) \cong \mathscr{H}_{v_0}(D, T) \times \Delta^n \cong E^n \times \Delta^n \cong E^{2n}$. This finishes the proof. □

The space $\mathscr{H}(D, T)$ is in general not pathwise connected even when (D, T) is a triangulated 3 dimensional simplex in E^3 (see [10]). Now suppose such a triangulated 3-simplex (D, T) in E^3 is given. Let v_0 be an interior vertex of T, $\mathscr{H}^0(D, T)$ be the path-component of $\mathscr{H}(D, T)$ that contains the identity map of D and let $\mathscr{H}^0_{v_0}(D, T) = \pi_{v_0}(\mathscr{H}^0(D, T))$.

Question. Is $\pi_{v_0}: \mathscr{H}^0(D, T) \to \mathscr{H}^0_{v_0}(D, T)$ always a trivial bundle?

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